

# The Fourier Splitting Method

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**Abstract:** In this paper we study the large time behaviour of solutions to some integral differential inequalities. These integral inequalities are satisfied by the solutions of several fluid equations as for example solutions to Navier-Stokes and Magneto Hydrodynamics equations. We show that the Fourier Splitting method can be used for the solutions of these inequalities to obtain algebraic energy decay rates. Moreover when the solutions are solutions to non linear differential equations ( as for example the mentioned above) the decay rate obtained is the same as the decay rate for their underlying linear counterpart.

## 1 Introduction

In this paper we study the large time behaviour of solutions to integral differential equations of the type

$$\frac{d}{dt} \int_{\mathbb{R}^n} |u|^2 dx \leq -C \int_{\mathbb{R}^n} |D^m u|^2 dx \quad (1)$$

The solutions of many fluid equations satisfy a relation of this type. For example solutions to Parabolic Conservation Laws, Navier-Stokes equations and Magneto Hydrodynamic equations satisfy (1) with  $m = 1$ . We will show that solutions of (1) which have some boundedness at the origin in frequency space will decay at a rate of order  $(t+1)^{-\frac{n}{2m}}$  in the  $L^2$  norm. More precisely we will show solutions to (1) have the same upper rate of decay as solutions to the linear equation

$$u_t = D^{2m} u.$$

The main tool needed to establish this decay is the Fourier splitting method first developed in [4, 5, 6] and then was extended in [9]. This method has been used for several diffusive equations all which satisfy (1), [6, 7, 8, 9, 3, 10]. This method does not depend on the linearized underlying equations. It mostly uses properties in Fourier space of the solutions. In what follows we will use the notation

$$S_m(t) = \left\{ \xi : |\xi| \leq \left( \frac{n}{C(t+1)} \right)^{\frac{1}{2m}} \right\} \quad (2)$$

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and

$$M_n = \{u : |\hat{u}(\xi, t)| \leq A_m \text{ for all } \xi \in S_m(t)\}, \quad (3)$$

where  $A_m$  is a positive constant. The main theorem in the paper deals with the rate of decay of weak solutions to inequalities of the type (1). More precisely

**Theorem 1:** *Let  $u_n(x, t)$  be a sequence of regular vector functions in  $\mathbb{R}^n$  with  $x \in \mathbb{R}^n$  and  $t \in \mathbb{R}_+$ . Suppose that the  $u_n$  converge weakly in  $L^2$  to  $u(x, t)$  and the  $u_n$ , for  $n$  large satisfy (1). Let  $u_n(x, t) = u_0 \in L^2$ . If there exists  $A_m$  such that  $|\hat{u}_n(\xi, t)| \in M_n$ , then*

$$\|u(\cdot, t)\|_{L^2}^2 \leq K(t+1)^{-\frac{2n}{m}}. \quad (4)$$

Here the constant  $K$  depends on the  $L^2$  norm of the data and the bound  $A_m$  of the  $L^\infty(S_m(t))$  norm of the Fourier Transform of the solution.

The second question we want to address is for what equations will the solutions  $u(x, t)$  satisfy an inequality of type (1) and be so that there exists a constant  $A_m$  such that  $u \in M_n$ .

We use the notation  $\|u\|_2$  for the  $L^2$  norm of  $u$  and  $\|u\|_\infty$  for the  $L^\infty$  norm of  $u$ .

## 2 The Upper Bounds

In this section we will establish Theorem 1 and follow it by a discussion on which class of functions will satisfy the hypothesis of the theorem. The proof of the theorem will be based on a straightforward application of the Fourier splitting method. Let us recall Theorem 1

**Theorem 1:** *Let  $u_n(x, t)$  be a sequence of regular vector functions in  $\mathbb{R}^n$  with  $x \in \mathbb{R}^n$  and  $t \in \mathbb{R}_+$ . Suppose that the  $u_n$  converge weakly in  $L^2$  to  $u(x, t)$  and the  $u_n$ , for  $n$  large satisfy (1). Let  $u_n(x, t) = u_0 \in L^2$ . If there exists  $A_m$  such that  $|\widehat{u}_n(\xi, t)| \in M_n$ , then*

$$\|u(\cdot, t)\|_{L^2}^2 \leq K(t+1)^{-\frac{2n}{m}}. \quad (5)$$

Where the constant  $K$  depends on the  $L^2$  norm of the data and the bound  $A_m$  of the  $L^\infty(S_m(t))$  norm of the Fourier Transform of the solution.

**Proof:** We will first show that the approximations  $u_n$  satisfy (5). Then passing to the limit the decay rate will follow for the limiting vector function  $u$ . For notation sake we will let  $u_n = u$ .

Using Plancherel's theorem on the energy relation (1)

$$\frac{d}{dt} \int_{\mathbb{R}^n} |u|^2 dx \leq -C \int_{\mathbb{R}^n} |D^m u|^2 dx,$$

it follows that

$$\frac{d}{dt} \int_{\mathbb{R}^n} |u|^2 dx \leq -C \int_{\mathbb{R}^n} |\xi^{2m}| |\hat{u}|^2 d\xi.$$

Let  $S_m = S_m(t)$  be defined by (2). Split the frequency space into two time dependent sets to get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} |u|^2 dx &\leq -C \int_{S_m} |\xi^{2m}| |\hat{u}|^2 d\xi - C \int_{S_m^c} |\xi^{2m}| |\hat{u}|^2 d\xi \\ &\leq -\frac{n}{t+1} \int_{S_m^c} |\hat{u}|^2 d\xi \\ &\leq -\frac{n}{t+1} \int_{\mathbb{R}^n} |\hat{u}|^2 d\xi + \frac{n}{t+1} \int_{S_m} |\hat{u}|^2 d\xi \end{aligned}$$

From the last inequality using  $(t + 1)^n$  as a multiplier, it follows that

$$\frac{d}{dt} (t + 1)^n \int_{\mathbb{R}^n} |u|^2 dx \leq n \left(\frac{n}{t+1}\right)^{n-1} \int_{S_m} |\hat{u}|^2 d\xi.$$

Hence

$$\frac{d}{dt} (t + 1)^n \int_{\mathbb{R}^n} |u|^2 dx \leq C(n, m) \left(\frac{n}{t+1}\right)^{n-1-\frac{n}{2m}},$$

where  $C(n, m) = nA_m^2 C^n \omega_0$  with  $\omega_0 =$  area of the  $n$ -dimensional sphere. Noting that  $n - 1 - \frac{n}{2m} \geq 0$ , since  $n \geq 2$  and  $m \geq 1$  if we integrate in time the last inequality and it follows that

$$(t + 1)^n \int_{\mathbb{R}^n} |u|^2 dx \leq C(n, m) \left[ n \frac{n}{2m} \left(\frac{n}{t+1}\right)^{n-\frac{n}{2m}} \right] + \int_{\mathbb{R}^n} |u_0|^2 dx.$$

Hence there is a constant  $K$  depending only the  $L^2$  norm of the data and the bound  $A_m$  of the  $L^\infty(S_m(t))$  norm of the Fourier Transform of the solution, such that inequality (5) holds for all  $u_n$  with  $n$  large. Passing to the limit and using standard analysis arguments will yield inequality (5) for the weak limit  $u$ . The proof of the theorem is complete.

Now we address the question in which cases is the Fourier Transform of  $u_n$  in the set  $M_m$ . One obvious answer is if the  $u_n \in \mathbf{L}^1$ . This is the case for scalar parabolic conservation laws in many variables [4]. A slightly weaker condition which also suffices is to have  $\hat{u}_n \in L^\infty$ . Moreover we actually only need to have  $u_n \in M_n$ . In the case where  $m = 1$  and  $n \geq 2$  the main examples are solutions to the Navier-Stokes and Magneto-Hydrodynamics equations, [4, 5, 6, 7, 9, 3] and [8]. Next consider the following scalar equation in  $n$  spatial variables:

$$u_t + \sum_{i=1}^n F_i(u)_{x_i} = D^{2m} u. \tag{6}$$

Here we use the notation  $G_{x_i} = \frac{\partial G}{\partial x_i}$  and hence  $F_i(u)_{x_i} = \frac{dF(u)}{du} \frac{\partial u}{\partial x_i}$ . When  $m = 1$  the scalar parabolic conservation Laws are recovered. The next theorem shows under which conditions solutions to (6) will be in  $M_n$ . We will consider only regular solutions. Similar results can be obtained for approximating solutions.

**Theorem 2:** Suppose  $u_0 \in L^2 \cap L^1 \cap C^m$ . Let  $u$  be a regular solution to (6) with data  $u_0$ . Where the  $F_i$  satisfy

$$|F_i(u)| \leq |u|^p. \tag{7}$$

If  $m = n(p - 1) + 1$  then  $u(x, t) \in M_n$ .

**Proof:** Take the Fourier Transform of (6) yields

$$\hat{u} + |\xi|^{2m} \hat{u} = \widehat{F_i(u)}.$$

Hence solving the O.D.E. yields

$$\hat{u} = \hat{u}_0 e^{|\xi|^{2m}(t-s)} + \int_0^t e^{-|\xi|^{2m}(t-s)} \sum_{i=1}^n \widehat{F_i(u)}_{x_i} ds = I_1 + I_2. \tag{8}$$

We note that the term  $I_1$  in (8) has an immediate bound since  $u_0 \in L^1$  by hypothesis. To bound  $I_2$  notice that it suffices to obtain the bound for just one of the terms in the sum since all the others will follow in the same way. For notation sake let  $F_i = F$ . We will show that

$$J = \int_0^t e^{-|\xi|^{2m}(t-s)} \widehat{F(u)}_{x_i}$$

is bounded. This will imply that  $I_2$  is also bounded. To bound  $J$  we use Schwartz' inequality and the Galiardo-Nirenberg inequality [1]. Since by hypothesis  $|F_{x_i}|$  grows like  $|u|^p$  it follows that

$$\begin{aligned} J &\leq \int_0^t |\widehat{F(u)}_{x_i}| ds \\ &\leq \int_0^t \int \left| \frac{dF}{du} u_{x_i} \right| dx ds + \int_0^t \left( \int_{\mathbb{R}^n} |u|^{2p} dx \right)^{1/2} \left( \int_{\mathbb{R}^n} |u_{x_i}|^2 dx \right)^{1/2} ds. \end{aligned}$$

From where it follows easily that

$$J \leq \int_0^t \|u\|_\infty^{2(p-1)} \int_{\mathbb{R}^n} |u|^2 dx \left( \int_{\mathbb{R}^n} |u_{x_i}|^2 dx \right)^{1/2} ds \tag{9}$$

From the Galiardo-Nirenberg inequality we have

$$\|u\|_\infty \leq c \|u\|_2^{1-a} \|D^m u\|_2^a,$$

with  $a = \frac{n}{2m}$  and also

$$\|Du\|_2 \leq c\|u\|_2^{1-b}\|D^m u\|_2^b,$$

with  $b = \frac{1}{m}$ .

Combining these specific cases of the Gagliardo-Nirenberg inequality with (9) yields

$$J \leq \int_0^t \left( \int_{\mathbb{R}^n} |u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |D^m u|^2 dx \right)^{\frac{n(p-1)+1}{m}} ds.$$

Recalling that the value of  $m$  in the hypothesis of the theorem was such that  $m = n(p - 1) + 1$ , it follows that

$$J \leq \int_0^t \left( \int_{\mathbb{R}^n} |u|^2 dx \right)^{\frac{1}{2}} \int_{\mathbb{R}^n} |D^m u|^2 dx. \tag{10}$$

We note that the  $L^2$  norm in space of  $u$  and the  $L^2$  norm in space and time of  $D^m u$  can be bounded easily. For this we multiply the (6) by  $u$  and integrate in space. After integrating by parts we get

$$\int_{\mathbb{R}^n} |u|^2 dx + \int_0^t \int_{\mathbb{R}^n} |D^m u|^2 dx ds \leq \int_{\mathbb{R}^n} |u_0|^2 dx. \tag{11}$$

From where the two bounds follow. Hence (10) and (11) yield

$$J \leq C \int_0^t \int_{\mathbb{R}^n} |D^m u|^2 dx ds$$

where the constant  $C$  depends only on  $\|u_0\|_2$ . From the last equation and (10) it follows that

$$J \leq C$$

here again  $C$  depends only on  $\|u_0\|_2$ . Since now we have that  $I_1$  and  $I_2$  are bounded it follows from (8) that  $u(x, t) \in M_n$ . The proof of Theorem 2 is now complete.

We finally would like to mention that the Fourier Splitting method also applies to systems of Parabolic Conservation Laws which admit a strictly convex entropy and the flux function has adequate polynomial growth [2].

### References

- [1] A. Friedman, *Partial Differential Equations*, Robert Krieger Publishing Co. (1983).
- [2] K. Khayat, dissertation (in preparation).
- [3] S.D. Mohgooner and R.E. Sarayker,  $L^2$  decay for solutions to the MHD equations, *Journal of Math. and Phys. Sciences* 23 No. 1 (1989), 35–53.
- [4] M.E. Schonbek, Decay of parabolic conservation laws, *Comm. in P.D.E.* 7 (1980), 449–473.

- [5] M.E. Schonbek,  $L^2$  decay of weak solutions of the Navier-Stokes equations, Arch. Rational Mech. Anal. **88** (1985), 209–222.
- [6] M.E. Schonbek, Large time behaviour of solutions of the Navier-Stokes equations, Comm. P.D.E. **11** (1986), 733–763 .
- [7] M.E. Schonbek, Lower bounds of rates of decay for solutions of the Navier-Stokes equations, J.A.M.S. **4** (1991), 423–449; Anal. **88** (1985), 209–222.
- [8] M.E. Schonbek , T.P. Schonbek, E.Suli, Large time behaviour of solutions to the Magneto-Hydrodynamics equations, preprint, submitted for publication.
- [9] M. Wiegner, Decay results for weak solutions to the Navier-Stokes equations in  $R^n$ , J. London Math. Soc. (2) **35** (1987), 303–313.
- [10] L. Zhang , Sharp rates of decay of global solutions to 2-dimensional Navier-Stokes equations, preprint.