MATH 205, HOMEWORK 4

Problem 1 (Chapter 2, Exercise 24). A *step function* is, by definition, a finite linear combination of characteristic functions of bounded intervals in $\mathbb{R}^1$. Assume $f \in L^1(\mathbb{R}^1)$, and prove that there is a sequence $\{g_n\}$ of step functions so that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x) - g_n(x)| \, dx = 0.$$

Answer. (Ruben) First note that it suffices to prove this result for f non-negative, because otherwise the negative and positive parts of f can be handled separately, and then put back together. Next note that there is an increasing sequence of simple functions h_n all less than such that $\int_{\mathbb{R}}(f(x) - h_n(x))\,dx \to 0$, because this is part of the definition of f being integrable. This means we will be done if we can approximate each h_n as a step function g_n arbitrarily well, in the sense that $\int_R |h_n - g_n|$ can be made as small as we want. Each h_n is the a finite linear combination of characteristic functions of sets of finite measure. Thus, to approximate h_n with g_n, it suffices to be able to approximate characteristic sets of measurable sets with characteristic functions of finite unions of bounded intervals.

Let E be a set of finite measure and let $\varepsilon > 0$. Then by regularity of Lebesgue measure, there is an open set $V \supseteq E$ such that $\mu(V \setminus E) < \varepsilon$. Open sets in $\mathbb{R}$ are the countable union of disjoint open intervals, in particular, $V = \bigcup_{k=1}^{\infty} I_k$. Because E is of finite measure, so is V, and so $\mu(V) = \sum_{k=1}^{\infty} \mu(I_k) < \infty$. Take N large enough that $\sum_{k=N+1}^{\infty} \mu(I_k) < \varepsilon$, and let $W = \bigcup_{k=1}^{N} I_k$. Calculate that

$$\int_{\mathbb{R}} |\chi_E - \chi_W| \, dx = \int_{\mathbb{R}} |\chi_E - \chi_V| \, dx + \int_{\mathbb{R}} |\chi_V - \chi_W| \, dx = \mu(V \setminus E) + \mu(V \setminus W) < \varepsilon.$$

□

Here is a more detailed answer:(Marta)

Proof. By the LDC, (since $|s_n| \le |f|$) we conclude that

$$\lim_n \int_{-\infty}^{\infty} s_n \, dx \;=\; \int_{-\infty}^{\infty} f \, dx. \; (1)$$

For each n, s_n is simple, hence, it is of the form $s_n = \sum_{i=1}^{k} a_{n,i}\chi_{E_{n,i}}$ (without loss of generality we will assume $a_{n,i} \neq 0$). We have that $m(E_{n,i}) < \infty$. Indeed,

$$m(E_{n,i}) \;=\; |a_{n,i}|^{-1} \int_{E_{n,i}} s_n \, dx \;\le\; |a_{n,i}|^{-1} \int_{-\infty}^{\infty} |f| \, dx \;<\; \infty.$$

Define $a_n := \sum_{i=1}^{k} a_{n,i}$.

We claim (see proof below) that for each $n > 0$ there exists $A_{n,i}$ such that

· $A_{n,i}$ is the finite disjoint union of open intervals ($A_{n,i} \;=\; \dot{\bigcup}_{m=1,\dots,l} I_{n,i}^m$)

Date: 5 February 2009.

· $m(E_{n,i} \triangle A_{n,i}) < \frac{1}{n}\frac{1}{ka_n}$.

Define $g_n := \sum_{i=1}^{k} a_{n,i}\ \chi_{A_{n,i}}$. Since

$$g_n := \sum_{i=1}^{k} a_{n,i}\ \chi_{A_{n,i}} \ = \ \sum_{i=1}^{k}\sum_{m=1}^{l} a_{n,i}\ \chi_{I^m_{n,i}},$$

g_n is a step function.
Moreover

$$\int_{-\infty}^{\infty} |f - g_n|\, dx \ \leq \ \underbrace{\int_{-\infty}^{\infty} |f - s_n|\, dx}_{\text{(by (1))}\longrightarrow 0} + \underbrace{\int_{-\infty}^{\infty} |s_n - g_n|\, dx}_{\longrightarrow 0 \text{ (by (*))}} \longrightarrow 0.$$

$$\begin{aligned}
(*) \quad \int_{-\infty}^{\infty} |s_n - g_n|\, dx \quad &= \quad \int_{-\infty}^{\infty} \Big|\sum_{i=1}^{k} a_{n,i}\ \chi_{E_{n,i}} - \sum_{i=1}^{k} a_{n,i}\ \chi_{A_{n,i}}\Big|\, dx \\
&\leq \quad \int_{-\infty}^{\infty} \sum_{i=1}^{k} |a_{n,i}|\ |\chi_{E_{n,i}} - \chi_{A_{n,i}}|\, dx \\
&= \quad \sum_{i=1}^{k} |a_{n,i}|\ m(E_{n,i} \triangle A_{n,i}) \ \leq \ a_n k \frac{1}{n}\frac{1}{k\ a_n} = \frac{1}{n} \longrightarrow 0.
\end{aligned}$$

Let $f \in L^1(\mu)$. Then $f = f^+ - f^-$. Since f^+ and f^- are positive functions, there exist two sequences of step functions, $\{g_n^+\}$ and $\{g_n^-\}$, such that

$$\lim_n \int_{-\infty}^{\infty} |f^+ - g_n^+|\, dx = 0$$

and

$$\lim_n \int_{-\infty}^{\infty} |f^- - g_n^-|\, dx = 0.$$

Define $g_n := g_n^+ - g_n^-$ (g_n is a step function).
Hence

$$\lim_n \int_{-\infty}^{\infty} |f - g_n|\, dx \leq \lim_n \int_{-\infty}^{\infty} |f^+ - g_n^+| \ + \ |f^- - g_n^-|\, dx = 0.$$

Proof of the claim: We will prove that if E is a measurable set with $m(E) < \infty$, then for every $\varepsilon > 0$ there is a set A that is the finite union of open intervals such that $m(E \triangle A) < \varepsilon$.

Let E be a measurable set with $m(E) < \infty$ and let $\varepsilon > 0$.
We have that

$$m(E) = \mathtt{inf}\ \{\sum_{j\in\mathbb{N}} \mu((a_j, b_j)) \mid E \subseteq \bigcup_{j\in\mathbb{N}} (a_j, b_j)\}.$$

and

$$m(E) = \mathtt{sup}\ \{\mu(K) \mid K \subseteq E \ \mathtt{and}\ K \ \mathtt{is\ compact}\}.$$

Let $\{(a_j, b_j)\}$ be a collection of open sets such that

- $E \subseteq \bigcup_{j\in\mathbb{N}} (a_j, b_j)$ and
- $m(E) + \frac{\varepsilon}{4} > \sum_{j\in\mathbb{N}} m((a_j, b_j))$.

Let K a compact set such that

- $K \subseteq E$ and
- $m(K) > m(E) - \frac{\varepsilon}{4}$.

Then $\{(a_j, b_j)\}$ covers K. Since K is compact, there is a finite subcover of $\{(a_j, b_j)\}$, $\{(a_{j_i}, b_{j_i})\}_{i=1}^m := U$.

Define $A := \bigcup_{i=1}^{m} (a_{j_i}, b_{j_i})$. Then

$$m(E \setminus A \cap E) \leq m(E \setminus K) < \frac{\varepsilon}{4}$$

(since $K \subseteq A \cap E$) and

$$m(A \setminus A \cap E) \leq m(A \setminus K) \leq m(U \setminus K)$$

(the first inequality holds because $K \subseteq A \cap E$ and the second one holds because $A \subseteq U$).

Since $U = E \cup (U \setminus E)$, then $U \setminus K = (E \setminus K) \cup (U \setminus E)$. Hence

$$m(U \setminus K) = m(E \setminus K) + m(U \setminus E) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Therefore,

$$m(E \triangle A) = m(E \setminus (A \cap E)) + m(A \setminus (A \cap E)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon.$$

□

Problem 2 (Chapter 3, Exercise 4). (Ruben + additions) Suppose f is a complex measurable function on X, μ a positive measure on X, and

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \quad (0 < p < \infty).$$

Let $E = \{p : \varphi(p) < \infty\}$. Assume $\|f\|_\infty > 0$.

a. If $r < p < s$, $r \in E$ and $s \in E$, prove that $p \in E$.

Answer. Define

$$f_{>1}(x) = \begin{cases} f(x) & \text{if } |f(x)| > 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_{\leq 1}(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then $f = f_{>1} + f_{\leq 1}$, and also $|f_{\leq 1}|^p \leq |f_{\leq 1}|^r$ and $|f_{>1}|^p \leq |f_{>1}|^s$, so

$$\begin{aligned} \int_X |f|^p \, d\mu &\leq \int_X |f_{\leq 1}|^r \, d\mu + \int_X |f_{>1}|^s \, d\mu \\ &\leq \int_X |f|^r \, d\mu + \int_X |f|^s \, d\mu < \infty. \end{aligned}$$

□

b. Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

Answer. If $r < s$ are in the interior of E, then $\varphi(r)$ and $\varphi(s)$ are finite and positive. Let $0 < \lambda < 1$, and set $p = \lambda r + (1-\lambda)s$. By Hölder's inequality,

$$\int_X |f|^p \, d\mu = \int_X |f|^{\lambda r} |f|^{(1-\lambda)s} \, d\mu \leq \left(\int_X |f|^r \, d\mu\right)^\lambda \left(\int_X |f|^s \, d\mu\right)^{1-\lambda}.$$

Thus, $\log(\varphi(p)) = \log(\varphi(\lambda r + (1-\lambda)s)) \leq \lambda \log(\varphi(r)) + (1-\lambda)\log(\varphi(s))$, and so $\log \varphi$ is convex, and therefore continuous on the interior of E. This means that φ is also continuous on the interior of E.

Addition to Ruben's solution For continuity of φ we have to prove : If $p \in E$ and $\{p_n\}$ is a sequence in E converging to $p \Rightarrow \ \varphi(p_n) \to \varphi(p)$. We may assume $\{p_n\}$ is either increasing or decreasing, (this is not essential it just simplifies the notation a bit). Set $q = \max(p.p_1), \ r = \min)p, p_1)$. The assumption on $\{p_n\}$ being monotone implies $r \leq p_n \leq q \, \forall n$. Let $A = \{x \in X : |f(x)| > 1\}$, $B = \{x \in X : |f(x)| \leq 1\}$,
Clearly $|f|^{p_n} \to |f|^p$ a.e; jn fact everywhere. Moreover,

$$|f|^{p_n} \leq |\chi_A|f|^q + \chi_B|f|^r.$$

The function$|\chi_A|f|^q + \chi_B|f|^r$. is integrable; the fact that $\{\varphi(p_n)\}$ converges to $\varphi(p)$ follows from Lebesgue's dominated convergence theorem. **End addition**

□

c. By a. E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$? **Here are several answers**

Answer 1. (Ruben)
E need not be open, or closed, and it is possible that E could be a single point.
For $\alpha > 0$, define functions

$$f_\alpha(x) = \begin{cases} x^{-1/\alpha} & \text{for } 0 < x \leq 1 \\ 0 & \text{for } 1 < x \end{cases} \quad \text{and} \quad g_\alpha(x) = \begin{cases} 0 & \text{for } 0 < x \leq 1 \\ x^{-1/\alpha} & \text{for } 1 < x \end{cases}.$$

Calculate that

$$\int_0^\infty f_\alpha(x)^p = \int_0^1 \frac{dx}{x^{p/\alpha}} \quad \text{and} \quad \int_0^\infty g_\alpha(x)^p = \int_1^\infty \frac{dx}{x^{p/\alpha}},$$

so $f_\alpha \in L^p$ if and only if $0 < p < \alpha$, and $g_\alpha \in L^p$ if and only if $\alpha < p < \infty$. Note also that $\|f_\alpha\|_p$ decreases monotonically and $\|g_\alpha\|_p$ as p increases while they remain finite.

Thus, the functions $f_{1+1/n}$ are all in L^p for $0 < p < 1 + 1/n$. By completeness of L^1, the series

$$\sum_{n=1}^\infty \frac{1}{2^n} \frac{f_{1+1/n}}{\|f_{1+1/n}\|_1}$$

converges to a function in $f \in L^1$, which is not in any L^p for $p > 1$. In fact $f \in L^p$ exactly when $0 < p \leq 1$.

On the other hand, the functions $g_{1+1/n}$ are all in L^p for $1+1/n < p$. By completeness of L^1, the series

$$\sum_{n=1}^\infty \frac{1}{2^n} \frac{g_{1+1/n}}{\|g_{1+1/n}\|_1}$$

converges to a function in $g \in L^1$, which is not in any L^p for $0 < p < 1$. In fact $g \in L^p$ exactly when $1 \leq p$.

Finally, note that $f + g \in L^1$, but not in L^p for any $p \neq 1$. □

Answer 2. Check out the function:

$$f(x) = \begin{cases} \frac{1}{x(\log(2/x))^2}, & 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

This will give you a function in L^1 and not in L^p for $p > 1$. Another function to play with is

$$f(x) = \begin{cases} \frac{1}{x(\log(2/x))^2}, & 0 < x \le 1 \\ \frac{1}{x(\log(x))^2}, & 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

This wil give a function in L^1 alone. □

Answer 3. (See in additional solutions)

□

d. If $r < p < s$, prove that $\|f\|_p \le \max(\|f\|_r, \|f\|_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subseteq L^p(\mu)$.

Answer. (Ruben) This follows from the convexity of $\log \varphi$. Write $p = \lambda r + (1-\lambda)s$, and calculate that

$$\begin{aligned} \log(\|f\|_p) = \tfrac{1}{p}\log(\varphi(p)) &\le \tfrac{1}{p}\big(\lambda \log(\varphi(r)) + (1-\lambda)\log(\varphi(s))\big) \\ &= \tfrac{1}{p}\big(\lambda r \log(\|f\|_r) + (1-\lambda)s\log(\|f\|_s)\big) \\ &\le \tfrac{1}{p}\big(\lambda r + (1-\lambda)s\big)\max\{\log(\|f\|_r), \log(\|f\|_s)\} \\ &= \max\{\log(\|f\|_r), \log(\|f\|_s)\} \end{aligned}$$

.

Thus, because exp is a monotonically increasing function, $\|f\|_p \le \max(\|f\|_r, \|f\|_s)$.
This implies that if $f \in L^r(\mu) \cap L^s(\mu)$ then $f \in L^p(\mu)$. □

e. Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that

$$\|f\|_p \to \|f\|_\infty \quad \text{as } p \to \infty.$$

Answer. (Rob) Part when $0 \le \|f\|_\infty < \infty$

Solution : If $\|f\|_\infty = 0$, then $\mu(X) = 0$ or $f = 0$ almost everywhere, and in either case $\|f\|_p = 0$ for all $p > 0$. Thus, assumine first that $\|f\|_\infty$ to be a positive real number β, (that is $\beta < \infty$)) we will show that $\lim_{p\to\infty} \|f\|_p \le \|f\|_\infty \le \lim_{p\to\infty} \|f\|_p$. To see the first inequality, we see that

$$\begin{aligned} \|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} &= \left(\int_X |f|^{p-r+r} \, d\mu\right)^{\frac{1}{p}} \\ &\le \left(\int_X \|f\|_\infty^{p-r} \cdot |f|^r \, d\mu\right)^{\frac{1}{p}} \\ &= (\|f\|_\infty)^{\frac{p-r}{p}} \cdot \left(\int_X |f|^r \, d\mu\right)^{\frac{1}{p}} \end{aligned}$$

Since $0 < \|f\|_r < \infty$, we see that $\lim_{p\to\infty}\left(\int_X \mid f \mid^r d\mu\right)^{\frac{1}{p}} = 1$ and so

$$\begin{aligned}\lim_{p\to\infty}\|f\|_p &\leq \lim_{p\to\infty}(\|f\|_\infty)^{\frac{p-r}{p}}\cdot\left(\int_X \mid f \mid^r d\mu\right)^{\frac{1}{p}}\\ &= \lim_{p\to\infty}(\|f\|_\infty)^{\frac{p-r}{p}}\cdot 1\\ &= \|f\|_\infty.\end{aligned}$$

To see the second inequality, we note that given any $\epsilon > 0$, the set $E = \{x \mid f(x) > \beta - \epsilon\}$ has positive measure. Then we see

$$\|f\|_p \geq \left(\int_E \mid f \mid^p d\mu\right)^{\frac{1}{p}} \geq \mu(E)^{\frac{1}{p}}\cdot(\beta - \epsilon)$$

Taking the limit as $p \to \infty$ we see that $\mu(E)^{\frac{1}{p}} \to 1$, and so we obtain

$$\lim_{p\to\infty}\|f\|_p \geq \beta - \epsilon$$

Since this inequality holds for all $\epsilon > 0$ we can say that $\lim_{p\to\infty}\|f\|_p \geq \|f\|_\infty$. Now since $\lim_{p\to\infty}\|f\|_p \leq \|f\|_\infty \leq \lim_{p\to\infty}\|f\|_p$, we can say that $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$.

Addition to Rob's answer Let $\|f\|_{infty} = \infty$. Let $A = n = \{x : |f(x)| > n\}$. Then we have that $A = \cup_{n=1}^{\infty} A_n$ has measure $\mu(A) > 0$ and $A_1 \supset A_2 \supset \ldots$.. We have two cases

- If there exists N so that $\mu(A_N) < \infty \Rightarrow \lim_{n\to\infty} = \mu(A)$.
- $\mu(A_n) = \infty\ \forall\ n$

Both cases yield $\lim_{n\to\infty}\|f\|_p = \infty$ since

$$n\mu(A_n)^{1/p} \leq \left(\int_{A_n} |f|^p dx\right)^{1/p} \quad \left(\int_X |f|^p dx\right)^{1/p}$$

The result follows letting $n \to \infty$. ☐

Problem 3 (Chapter 3, Exercise 5). Assume, in addition to the hypotheses of Exercise 4, that $\mu(X) = 1$.

a. Prove that $\|f\|_r \leq \|f\|_s$ if $0 < r < s \leq \infty$.

Answer. Let $t = s/r > 0$ and v be conjugate to t. Then by Hölder's inequality, $\|f^r\|_1 \leq \|f^r\|_t\|1\|_v = \|f^r\|_t$, because $\mu(X) = 1$. Rewrite in terms of r and s to get

$$\|f\|_r = \|f^r\|_1^{1/r} \leq \|f^r\|_t^{1/r} = \left(\left(\int_x (|f|^r)^t\, d\mu\right)^{1/t}\right)^{1/r}$$

$$= \left(\int_x |f|^s\, d\mu\right)^{1/s} = \|f\|_s.$$

☐

b. Under what conditions does it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s < \infty$?

Answer.](Ruben) It is clear that one possible condition under which $\|f\|_r = \|f\|_s < \infty$ is for f being almost everywhere constant. This also the only condition under which this can happen.

Let f be such that $\|f\|_r = \|f\|_s < \infty$. We can assume that $\|f\|_r = 1$, because otherwise we could scale f to make that true. Let $g = |f|^r$, and $t = s/r$. Then $\|g\|_1 = \|f\|_r^r = 1$ and $\|g\|_t^t = \int |f|^{rt}\, d\mu = \int |f|^s\, d\mu = 1$.

Let v be the exponent conjugate to t. By Hölder's inequality, $1 = \|g\|_1 \leq \|1\|_v \|g\|_t = 1 \cdot 1$. This completes the proof, because the only conditions under which equality hold for Hölder's inequality applied to two positive functions are when the two functions involved are scalar multiplies of each other. In this case, that would make g a scalar multiple of 1. □

c. Prove that $L^r(\mu) \supseteq L^s(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

Answer.](Ruben) $L^r(\mu) \supseteq L^s(\mu)$ if $0 < r < s$ follows from part a.

It happens if μ is a measure over a finite σ-algebra. This may be the only conditions under which this happens, but I do not have time to investigate now. □

d. Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p \to 0} \|f\|_p = \exp\left(\int_X \log|f|\, d\mu\right)$$

if $\exp(-\infty)$ is defined to be 0.

Answer. (Marta; solution when $\log|f| \in L^1$)

Assume that $||f||_r < \infty$ for some $r > 0$. and $\log|f| \in L^1$

Since `exp` is a convex function and $log \circ |f|^p$ is in $L^1(\mu)$, by Jensen's inequality, we have that

$$\cdot \quad \left(\texttt{exp}\left\{\int_X log \circ |f|^p\, d\mu\right\}\right)^{\frac{1}{p}} \leq \left(\int_X \texttt{exp} \circ log \circ |f|^p\, d\mu\right)^{\frac{1}{p}} \Leftrightarrow$$

$$\Leftrightarrow \quad \left(\texttt{exp}\left\{p\int_X log \circ |f|\, d\mu\right\}\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}} \Leftrightarrow$$

$$\Leftrightarrow \quad \texttt{exp}\left\{\int_X log \circ |f|\, d\mu\right\} \leq ||f||_p.$$

We have that (for small p) $\frac{|f|^p - 1}{p} \leq 2\log|f|$ since $\log|f| \in L^1$ by LDC,

$$\lim_{p\to 0}\int_X \frac{|f|^p - 1}{p}\, d\mu \;=\; \int_X \lim_{p\to 0}\frac{|f|^p - 1}{p}.$$

Since $log(u) \;\leq\; u - 1$ (for $u \geq 0$ defining $log(0) := -\infty$), we have

$$\begin{aligned} log(||f||_p) &= \frac{1}{p}\, log\left(\int_X |f|^p\, d\mu\right) \\ &\leq \frac{1}{p}\left(\int_X |f|^p\, d\mu \;-\; 1\right) \\ &= \frac{1}{p}\left(\int_X |f|^p\, d\mu \;-\; \mu(X)\right) \\ &= \frac{1}{p}\int_X (|f|^p - 1)\, d\mu \\ &= \int_X \frac{|f|^p - 1}{p}\, d\mu. \end{aligned}$$

Then,

$$\begin{aligned}\lim_{p\to 0} log(||f||_p) &\leq \lim_{p\to 0}\int_X \frac{|f|^p-1}{p}\, d\mu\\ &= \int_X \lim_{p\to 0}\frac{|f|^p-1}{p}\, d\mu\\ &= \int_X log|f|\, d\mu\end{aligned}$$

and, hence (since exp is continuous),

$$\lim_{p\to 0}||f||_p \leq \texttt{exp}\left(\int_X log|f|\, d\mu\right).$$

And case when $\log|f| \in L^1$ follows.

Addition to Marta's solution

Case 2. $\log|f| \notin L^1$

In this case since $\texttt{exp}\left\{\int_X log|f|\, d\mu\right\} \leq ||f||_p$, it follows that

$$\int_X log|f|\, d\mu = -\infty$$

since $\|f\|^p < \infty$. Hence we need to show that $\|f\|_p \to 0$ as $p \to 0$. This can be done easily if f is a characteristic function. Note that for $\int_X log|\chi_E|d\mu = -\infty$ the set $E \subset X$ and $e \neq X$. Hence $\|\chi_E\|_p = \mu(E)^{1/p} \to 0,$ when $p \to 0$ since$\mu(E) < 1$. The analysis for simple functions is the same. Recall $f \in L^r$ some $r > 0$, we can take $r, 1$. Let s_n be simple functions such that,

$$0 \leq s_1 \leq s_2 \leq .... \leq f, \text{ and } s_n(s) \uparrow f(x),\ \forall x \in X$$

For any $p \in (0,1)$ one has the reverse Minkoswski inequality

$$\|f+g\|_p \geq \|f\|_p + \|g\|p$$

let $g = s_n - f$ then

$$\|f\|_p \geq \|f - s_n\|_p + \|s_n\|p \qquad (1)$$

Let $p < r$ then $r/p > 1$ and we can use Jensen's inequality. to obtain

$$\|f - s_n\|_p = \left(\int_X |f - s_n|^p d\mu\right)^{(r/p)(1/r)} \leq \|f\|_r \quad ;\ (2)$$

Since $f \geq s_n$ it follows that $\|f - s_n\|_r \|f\|_r < \infty$. Hence By LDC (and Jensen) it follows that

$$\lim_{n\to\infty}\|f - s_n\|_p \leq \lim_{n\to infty}\|f - s_n\|_r = \int \lim_{n\to infty}|f - s_n|d\mu = 0$$

Therefore given $\epsilon > 0$ there exists N_r so that $\forall\ n \geq N_r$

$$\|f - s_n\|_p \leq \frac{\epsilon}{2} \qquad (3)$$

Since for all n we had $\|s_n\|_p \to 0,$ as$p \to 0$, we have that there is p_o so that for all $p \geq P_o$ it follows that

$$\|s_{N_r}\| \leq \frac{\epsilon}{2} \qquad (4)$$

Combining (1), (3) and (4) yields for all $p \geq P_o$

$$\|f\|_p \le \|s_{N_r}\|_p + \|f - s_{N_r}\|_p \le \epsilon$$

Since ϵ was arbitrary the result follows

□

Problem 4 (Chapter 3, Exercise 9). (Ruben)
Suppose that f is Lebesgue measurable on $(0,1)$ and not essentially bounded. By Exercise 4(e), $\|f\|_p \to \infty$ as $p \to \infty$. Can $\|f\|_p$ tend to ∞ arbitrarily slowly? More precisely, is it true that to every positive function Φ on $(0,\infty)$ such that $\Phi(p) \to \infty$ as $p \to \infty$ one can find an f such that $\|f\|_p \to \infty$ as $p \to \infty$ for all sufficiently large p?

Answer. Calculate that

$$\|x^\alpha\|_p = \left(\frac{1}{\alpha p}\right)^{1/p},$$

so

$$\left\|\sum_{\alpha\in I} x^\alpha\right\|_p \le \sum_{\alpha\in I}\left(\frac{1}{\alpha p}\right)^{1/p} = \left(\frac{1}{p}\right)^{1/p}\sum_{\alpha\in I}\left(\frac{1}{\alpha}\right)^{1/p}$$
$$= \sum_{\alpha\in I}\left(\frac{1}{\alpha}\right)^{1/p}$$

For every $n = 1, 2, \ldots,$ there is a p_n such that $\Phi(p) > n+1$ for $p > p_n$. For each n, choose α_n large enough that $(1/\alpha_n)^{1/p} < 2^{-n}$ for $p > p_n$, and let I be the set of these chosen α_n's. Define $f(x) = \sum_{\alpha\in I} x^\alpha$, and calculate that for $p < p_n$,

$$\|f\|_p \le \sum_{k=1}^{n}\left(\frac{1}{\alpha_k}\right)^{1/p} + \sum_{k=n+1}^{\infty}\left(\frac{1}{\alpha_k}\right)^{1/p}$$
$$< n + \sum_{k=n+1}^{\infty} 2^{-k} < n+1.$$

□

See additional solution

Problem 5 (Chapter 3, Exercise 10). (Ruben)
Suppose $f_n \in L^p(\mu)$, for $n = 1, 2, 3, \ldots,$ and $\|f_n - f\|_p \to 0$ and $f_n \to g$ a. e., as $n \to \infty$. What relation exists between f and g.

Answer. $f = g$ a. e., because $\|f_n - f\|_p \to 0$ implies that $f_n \to f$ a. e. □

Problem 6 (Chapter 3, Exercise 11). (Ruben) Suppose $\mu(\Omega) = 1$ and suppose f and g are positive measurable functions on Ω such that $fg \ge 1$. Prove that

$$\int_\Omega f\,d\mu \cdot \int_\Omega g\,d\mu \ge 1.$$

Answer. $f^{1/2}g^{1/2} \ge 1$, so by Schwarz's inequality

$$1 \le \int_\Omega f^{1/2}g^{1/2}\,d\mu \le \left(\int_\Omega f\,d\mu\right)^2\left(\int_\Omega g\,d\mu\right)^2.$$

Take the square root of both sides to get

$$\int_\Omega f\,d\mu \cdot \int_\Omega g\,d\mu \geq 1.$$

□

Problem 7 (Chapter 3, Exercise 12). (Ruben) Suppose $\mu(\Omega) = 1$ and $h : \Omega \to [0, \infty]$ is measurable. If

$$A = \int_\Omega h\,d\mu,$$

prove that

$$\sqrt{1+A} \leq \int_\Omega \sqrt{1+h^2}\,d\mu \leq 1 + A.$$

If μ is Lebesgue measure on $[0,1]$ and if f is continuous, $h = f'$, the above inequalities have a simple geometric interpretation. From this, conjecture (for general Ω) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

Answer. The function $\sqrt{1+x^2}$ has a positive second derivative, and is therefore convex. Thus Jensen's inequality gives $\sqrt{1+A} \leq \int_\Omega \sqrt{1+h^2}\,d\mu$. Also, note that $\sqrt{1+x^2} \leq 1+x$ for all $x \geq 0$, because its first derivative, $x/\sqrt{1+x^2}$ is bounded by 1. Hence,

$$\int_\Omega \sqrt{1+h^2}\,d\mu \leq \int_\Omega (1+h)\,d\mu \leq 1 + A.$$

The second inequality will be strict only if h is zero almost everywhere, because $\sqrt{1+x^2} < 1+x$ for $x > 0$. The first inequality will hold only for h constant, because the function $\sqrt{1+x^2}$ is strictly convex for $x > 0$. □

See in additional solutions the answer for the geometric part