Some Practice Problems for the Final

These problems are generally a little bit trickier than what you'll see on the final, but are tractable and should be good practice.

1. Let C be a regular curve with everywhere nonzero curvature and let $\alpha(t)$ be a parametrization of C by arc length. Define the evolute β of α by $\beta(t) := \alpha(t) + \frac{1}{k(t)}\mathbf{n}(t)$, where $k(t)$ denotes the curvature of α.

   (a) Under what conditions does $\beta(t)$ determine a regular parametrized curve?

   (b) Define $f(t) := \beta(t)\cdot\alpha'(t)$. Given t_0 in the domain of α, compute $f'(t_0)$ and $f''(t_0)$ in terms of the Frenet frame $\{\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0\}$ of α at t_0 and the values of the curvature and torsion functions and their derivatives at t_0.

2. Let S_1 and S_2 be regular surfaces that intersect along a regular curve C with everywhere nonzero curvature. Suppose that $T_pS_1 \neq T_pS_2$ for every point $\mathbf{p} \in C$. Show that the Frenet frames along C can be expressed up to an overall choice of sign for the tangent $\mathbf{t}_C$ and binormal $\mathbf{b}_C$ in terms of the normals $\mathbf{n}_j$ of S_j and normal curvatures $k_{C,j}$ of C with respect to S_j, $j = 1, 2$, *without* using a parametrization of C.
   Hint: Show that $\{\mathbf{t}_C(\mathbf{p}), \mathbf{n}_1(\mathbf{p}), \mathbf{n}_2(\mathbf{p})\}$ is a basis of $\mathbb{R}^3$ for each $\mathbf{p} \in C$, and then solve for the coefficients for the normal $\mathbf{n}_C(\mathbf{p})$ of C at $\mathbf{p}$ in terms of that basis, expressing those coefficients in terms of the normal curvatures and appropriate combinations of the surface normals. Don't write out a detailed expression for the binormal — just express it in terms of the tangent and normal vectors.

3. (Lagrange multipliers) Let $S = g^{-1}(c)$ be the level set of a regular value c of a smooth function $g : \mathbb{R}^3 \to \mathbb{R}$. Let $f|_S$ denote the restriction to S of another smooth function $f : \mathbb{R}^3 \to \mathbb{R}$. Show that $p \in S$ is a critical point of $F|_S$, i.e. $d(f|_S)_p = 0$, if and only if $\nabla f(p) = \lambda \nabla g(p)$ for some $\lambda \in \mathbb{R}$.

4. (a) Let S be a regular surface, $\mathbf{p} \in S$, and $\{\mathbf{e}_1, \mathbf{e}_2\}$ be a basis for T_pS. Let $L : T_pS \to T_pS$ be a symmetric linear transformation with matrix representation A with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. Show that for any vectors $\mathbf{v}$ and $\mathbf{w}$ in T_pS

   $$(L\mathbf{v}) \times (L\mathbf{w}) = (\det A)\, \mathbf{v} \times \mathbf{w} \qquad \text{and} \qquad (L\mathbf{v}) \times \mathbf{w} + \mathbf{v} \times (L\mathbf{w}) = (\operatorname{trace} A)\, \mathbf{v} \times \mathbf{w}.$$

   Hint: First argue that, due to the bilinearity and antisymmetry of the cross product, it suffices to consider only the case $\mathbf{v} = \mathbf{e}_1$, $\mathbf{w} = \mathbf{e}_2$.

   (b) Show that part (a) implies that

   $$(d\mathbf{n}_p\, \mathbf{v}) \times (d\mathbf{n}_p\, \mathbf{w}) = K(\mathbf{p})\, \mathbf{v} \times \mathbf{w} \qquad \text{and} \qquad (d\mathbf{n}_p\, \mathbf{v}) \times \mathbf{w} + \mathbf{v} \times (d\mathbf{n}_p\, \mathbf{w}) = 2H(\mathbf{p})\, \mathbf{v} \times \mathbf{w},$$

   where $K(\mathbf{p})$ denotes the Gaussian curvature and $H(\mathbf{p})$ the mean curvature of S at $\mathbf{p}$.

5. Give an example of each of the following kinds of maps:

   (a) A diffeomorphism from a regular surface containing only elliptic points to a regular surface containing only parabolic points.

   (b) A diffeomorphism of a regular surface onto itself such that

      i. any elliptic and planar points are mapped to hyperbolic points
      ii. any hyperbolic points are mapped to elliptic points
      iii. any parabolic points are mapped to parabolic points

   You do not need to give explicit formulas for these surfaces and maps, but you should describe them clearly and justify your answers.