Arnold, ODEs

30. Contraction Mappings

We now give a method for finding a fixed point of a mapping of a metric space into itself. This method will be used later to construct solutions of differential equations.

30.1. Definition. Let $A: M \to M$ be a mapping of a metric space M (with metric ρ) into itself. Then M is said to be a *contraction mapping* if there exists a constant λ, $0 < \lambda < 1$ such that

$$\rho(Ax, Ay) \leqslant \lambda\rho(x, y) \quad \forall\, x, y \in M. \tag{1}$$

Example 1. Let $A: \mathbf{R} \to \mathbf{R}$ be a real function of a real variable (Fig. 198). If the derivative of A is everywhere of absolute value less than 1, then A need not be a contraction mapping. However A is a contraction mapping if

$|A'| \leqslant \lambda < 1.$

Example 2. Let $A: \mathbf{R}^n \to \mathbf{R}^n$ be a linear operator. If all the eigenvalues of A lie strictly inside the unit disk, then there exists a Euclidean metric (a Lyapunov function in the sense of Sec. 22.3) such that A is a contraction mapping.

Problem 1. Which of the following mappings of the line (with the ordinary metric) into itself are contraction mappings:

a) $y = \sin x$; b) $y = \sqrt{x^2 + 1}$; c) $y = \arctan x$?

Problem 2. Can $\leqslant$ be replaced by $<$ in the inequality (1)?

For, eg., $\mathbb{R}^n$, $\rho(x,y) = \|x - y\|$, so $A: \mathbb{R}^n \to \mathbb{R}^n$ is a contraction mapping if $\|A(x) - A(y)\| \leq \lambda \|x - y\|$ $\forall\, x, y \in \mathbb{R}^n$ etc.

30.2. The contraction mapping theorem. A point $x \in M$ is called a *fixed point* of the mapping $A: M \to M$ if $Ax = x$.

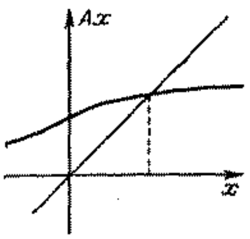

Fig. 198 Fixed point of a contraction mapping.

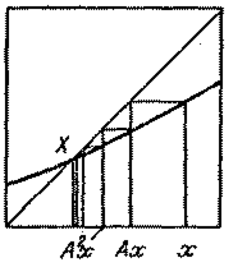

Fig. 199 Sequence of images of a point x under a mapping A.

THEOREM. *Let $A: M \to M$ be a contraction mapping of a complete metric space M into itself. Then A has a unique fixed point. Given any point $x \in M$, the sequence*

$x, Ax, A^2x, A^3x, \ldots$

of images of x under application of the operator A (Fig. 199) *converges to the fixed point.*

Proof. If $\rho(x, Ax) = d$, then

$$\rho(A^n x, A^{n+1}x) \leqslant \lambda^n d.$$

The series

$$\sum_{n=0}^{\infty} \lambda^n$$

converges, and hence the sequence $A^n x$, $n = 0, 1, 2, \ldots$ is a Cauchy sequence. But the space M is complete, and hence the limit

$$X = \lim_{n \to \infty} A^n x$$

exists. The point X is a fixed point of A. In fact, since every contraction mapping is continuous (choose $\delta = \varepsilon$), we have

$$AX = A \lim_{n \to \infty} A^n x = \lim_{n \to \infty} A^{n+1} x = X.$$

Moreover every fixed point Y coincides with X, since

$$\rho(X, Y) = \rho(AX, AY) \leqslant \lambda\rho(X, Y),\ \lambda < 1 \Rightarrow \rho(X, Y) = 0. \ ∎$$

A sequence $\{x_n\}$ is Cauchy if for any $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ such that $\rho(x_k, x_n) < \varepsilon$ if $k, n \geq N$.

M is complete if every Cauchy seq has a limit in M.

Remark. The points $x, Ax, A^2x, \ldots$ are called *successive approximations* to X.

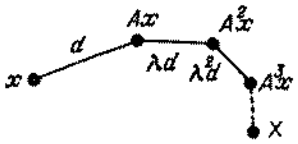

Fig. 200 Estimate of the accuracy of an approximation x to the fixed point X.

Let x be an approximation to the fixed point X of a contraction mapping A. Then the accuracy of the approximation is easily estimated in terms of the distance d between the points x and Ax. In fact

$$\rho(x, X) \leqslant \frac{d}{1-\lambda},$$

since

$$d + \lambda d + \lambda^2 d + \cdots = \frac{d}{1-\lambda}$$

(Fig. 200).

Problem 1. Prove the inverse function theorem.

Hint. It is sufficient to invert a C^1-mapping with a unit linear part $\mathbf{y} = \mathbf{x} + \boldsymbol{\varphi}(\mathbf{x})$, where $\boldsymbol{\varphi}'(0) = 0$ in a neighborhood of the point $0 \in \mathbf{R}^n$ (a linear change of variables reduces the general case to this case). Suppose we look for a solution of the form $\mathbf{x} = \mathbf{y} + \boldsymbol{\psi}(\mathbf{y})$. Then we get the equation

$$\boldsymbol{\psi}(\mathbf{y}) = -\boldsymbol{\varphi}(\mathbf{y} + \boldsymbol{\psi}(\mathbf{y}))$$

for $\boldsymbol{\psi}$. Therefore the desired function $\boldsymbol{\psi}$ is a fixed point of the mapping A defined by the formula

$$(A\boldsymbol{\psi})(\mathbf{y}) = -\boldsymbol{\varphi}(\mathbf{y} + \boldsymbol{\psi}(\mathbf{y})).$$

Moreover A is a contraction mapping (in a suitable metric), since the derivative of the function $\boldsymbol{\varphi}$ is small in a neighborhood of the point 0 (because of the condition $\boldsymbol{\varphi}'(0) = 0$).

The space $C^1(D)$ of fcns* with continuous first derivatives has the metric

$$(\rho(f,g))^2 = \max_{\underset{\sim}{x} \in D} \left(\|\underset{\sim}{f}(\underset{\sim}{x}) - \underset{\sim}{g}(\underset{\sim}{x})\|^2 + \|\underset{\sim}{f}'(\underset{\sim}{x}) - \underset{\sim}{g}'(\underset{\sim}{x})\|^2 \right)$$

(*) fcns $\underset{\sim}{f}: D \to \mathbb{R}^n$ for some compact set $D \subseteq \mathbb{R}^\ell$.

All sorts of other metric function spaces are out there; many, but not all, are complete. This sort of stuff is central to functional analysis.